

HEAT KERNEL AND CURVATURE BOUNDS IN RICCI FLOWS WITH BOUNDED SCALAR CURVATURE — PART II

RICHARD H. BAMLER AND QI S. ZHANG

ABSTRACT. In this paper we analyze the behavior of the distance function under Ricci flows whose scalar curvature is uniformly bounded. We will show that on small time-intervals the distance function is $\frac{1}{2}$ -Hölder continuous in a uniform sense. This implies that the distance function can be extended continuously up to the singular time.

1. INTRODUCTION

In this paper, we extend the estimates of [BZ15], to prove the following result:

Theorem 1.1. *For any $0 < A < \infty$ and $n \in \mathbb{N}$ there is a constant $C = C(A, n) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ be a Ricci flow ($\partial_t g_t = -2\text{Ric}_{g_t}$) on an n -dimensional compact manifold \mathbf{M} with the property that $\nu[g_0, 1 + A^{-1}] \geq -A$. Assume that the scalar curvature satisfies $|R| \leq R_0$ on $\mathbf{M} \times [0, 1]$ for some constant $0 \leq R_0 \leq A$.

Then for any $0 \leq t_1 \leq t_2 \leq 1$ and $x, y \in \mathbf{M}$ we have the distance bound

$$d_{t_1}(x, y) - C\sqrt{t_2 - t_1} \leq d_{t_2}(x, y) \leq \exp(CR_0^{1/2}\sqrt{t_2 - t_1})d_{t_1}(x, y) + C\sqrt{t_2 - t_1}.$$

In particular, if $\min\{d_{t_1}(x, y), d_{t_2}(x, y)\} \leq D$ for some $D < \infty$, then

$$|d_{t_1}(x, y) - d_{t_2}(x, y)| \leq C'\sqrt{t_2 - t_1},$$

where C' may depend on A, D and n .

By parabolic rescaling, we obtain distance bounds on larger time-intervals. Note that Theorem 1.1 is a generalization of [BZ15, Theorem 1.1], which only provides a bound on the distance distortion that does not improve for t_2 close to t_1 . The constant $\nu[g_0, 1 + A^{-1}]$ is defined as the infimum of Perelman's μ -functional (cf [Per02]) $\mu[g_0, \tau]$ over all $\tau \in (0, 1 + A^{-1})$. For more details see [BZ15, sec 2]. The condition $\nu[g_0, 1 + A^{-1}] \geq -A$, can be viewed as a non-collapsing condition. The exponential factor in the upper bound is necessary, as one can see for example in the case in which $(\mathbf{M}, (g_t)_{t \in [0,1]})$ is the Ricci flow on a hyperbolic manifold and the distance between x, y is very large. The proof of Theorem 1.1 will heavily use the results of [BZ15], in particular the heat kernel bound, [BZ15, Theorem 1.4].

As a consequence of Theorem 1.1, we obtain the following:

Corollary 1.2. *Let $(\mathbf{M}, (g_t)_{t \in [0, T]})$, $T < \infty$ be a Ricci flow on a compact manifold and assume that the scalar curvature satisfies $R < C < \infty$ on $\mathbf{M} \times [0, T]$. Then the distance function*

$$d : \mathbf{M} \times \mathbf{M} \times [0, T] \longrightarrow [0, \infty), \quad (x, y, t) \longmapsto d_t(x, y)$$

can be extended continuously onto the domain $\mathbf{M} \times \mathbf{M} \times [0, T]$.

Note that the corollary does not state that $d_T : \mathbf{M} \times \mathbf{M} \rightarrow [0, \infty)$ is a metric on \mathbf{M} . It only follows that d_T is a pseudometric, which means that we may have $d_T(x, y) = 0$ for some $x \neq y$. After taking the metric identification, however, $(\mathbf{M}/\sim, d_T)$ is in fact the Gromov-Hausdorff limit of (\mathbf{M}, g_t) as $t \nearrow T$. Here $x \sim y$ if and only if $d_T(x, y) = 0$. Moreover, since the volume measure converges as well, the space $(\mathbf{M}/\sim, d_T)$ becomes a metric measure space with doubling property and this space is the limit of (\mathbf{M}, g_t) in the measured Gromov-Hausdorff sense.

More generally, we obtain the following consequence of Theorem 1.1.

Corollary 1.3. *Let $(\mathbf{M}^i, (g_t^i)_{t \in [0, 1]})$ be a sequence of Ricci flows on n -dimensional compact manifolds \mathbf{M}^i with the property that $\nu[g_0^i, 1 + A^{-1}] \geq -A$ and $|R| < A$ on $\mathbf{M} \times [0, 1]$ for some uniform $A < \infty$. Let $x_i \in \mathbf{M}^i$ be points. Then, after passing to a subsequence, we can find a pointed metric space $(\overline{\mathbf{M}}, \overline{d}, \overline{x})$, a continuous function*

$$d^\infty : \overline{\mathbf{M}} \times \overline{\mathbf{M}} \times [0, 1] \rightarrow [0, \infty), \quad (x, y, t) \mapsto d_t^\infty(x, y)$$

and a continuous family of measures $(\mu_t)_{t \in [0, 1]}$ such that for any $x, y \in \overline{\mathbf{M}}$, the function $t \mapsto d_t^\infty(x, y)$ is $\frac{1}{2}$ -Hölder continuous and such that for any $t \in [0, 1]$, the metric identification $(\overline{\mathbf{M}}/\sim_t, d_t^\infty, \mu_t, \overline{x})$ is a metric measure space with doubling property for balls of radius less than \sqrt{t} . Here $x \sim_t y$ if and only if $d_t^\infty(x, y) = 0$. Moreover, for any $t \in [0, 1]$ the sequence $(\mathbf{M}^i, g_t^i, dg_t^i, x_i)$ converges to $(\overline{\mathbf{M}}/\sim_t, d_t^\infty, \mu_t, \overline{x})$ in the pointed, measured Gromov-Hausdorff sense.

For the proof of Corollary 1.3 see section 5.

Note that if we impose the extra assumption that $|R| < R_i$ on $\mathbf{M} \times [0, 1]$ for some sequence R_i with $\lim_{i \rightarrow \infty} R_i = 0$, then the limiting family of measures $(\mu_t)_{t \in [0, 1]}$ is constant in time. Unfortunately, however, our results do not imply that $(d_t^\infty)_{t \in [0, 1]}$ is constant in time as well.

Finally, we mention a direct consequence of Theorem 1.1, which can be interpreted as an analogue of the main result of [CN12] in the parabolic case.

Corollary 1.4. *For any $0 < A < \infty$ and $n \in \mathbb{N}$ there is a constant $C = C(A, n) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0, 1]})$ be a Ricci flow on an n -dimensional compact manifold \mathbf{M} with the property that $\nu[g_0, 1 + A^{-1}] \geq -A$. Assume that the scalar curvature satisfies $|R| \leq A$ on $\mathbf{M} \times [0, 1]$.

Then for any $r > 0$ and $0 \leq t_1 \leq t_2 \leq 1$ and $x \in \mathbf{M}$ we have the following bound for Gromov-Hausdorff distance of r -balls

$$d_{\text{GH}}(B(x, t_1, r), B(x, t_2, r)) \leq C\sqrt{|t_1 - t_2|}.$$

For the rest of the paper, we will fix the dimension $n \geq 2$ of the manifold \mathbf{M} . Most of our constants will depend on n . For convenience we will not mention this dependence anymore.

2. UPPER VOLUME BOUND

We first generalize the upper volume bound from [Zha12] or [CW13].

Lemma 2.1. *For any $A < \infty$ there is a uniform constant $C_0 = C_0(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Assume that $\nu[g_{-1}, 4] \geq -A$. Then for any $(x, t) \in \mathbf{M} \times [0, 1]$ and $r > 0$ we have

$$|B(x, t, r)|_t < C_0 r^n e^{C_0 r}.$$

Here $|S|_t$ denotes the volume of a set $S \subset \mathbf{M}$ with respect to the metric g_t .

Proof. It follows from [Per02], [Zha12], [CW13] (see also [BZ15, sec 2]), that for any $x \in \mathbf{M}$ and $0 \leq r \leq 1$, we have

$$(2.1) \quad cr^n \leq |B(x, t_0, r)|_{t_0} \leq Cr^n,$$

for some constants c, C , which only depend on A .

Fix some $x \in \mathbf{M}$ and let $N < \infty$ be maximal with the property that we can find points $x_1, \dots, x_N \in B(x, t, \frac{1}{2})$ such that the balls $B(x_1, t, \frac{1}{8}), \dots, B(x_N, t, \frac{1}{8})$ are pairwise disjoint. Note that then

$$B(x_1, t, \frac{1}{8}), \dots, B(x_N, t, \frac{1}{8}) \subset B(x, t, 1).$$

So, by (2.1), we have $N \leq C_* := (c(\frac{1}{8})^n)^{-1}C$. Moreover, by the maximality of N , we have

$$(2.2) \quad B(x_1, t, \frac{1}{4}) \cup \dots \cup B(x_N, t, \frac{1}{4}) \supset B(x, t, \frac{1}{2}).$$

We now argue that for all $r \geq \frac{1}{2}$

$$(2.3) \quad B(x_1, t, r) \cup \dots \cup B(x_N, t, r) \supset B(x, t, r + \frac{1}{4}).$$

Let $y \in B(x, t, r + \frac{1}{4})$ and consider a time- t minimizing geodesic $\gamma : [0, l] \rightarrow \mathbf{M}$ between x and y that is parameterized by arclength. Then $l < r + \frac{1}{4}$. By (2.2) we may pick $i \in \{1, \dots, N\}$ such that $\gamma(\frac{1}{2}) \in \overline{B(x_i, t, \frac{1}{4})}$. Then

$$\text{dist}_t(x_i, y) \leq (l - \frac{1}{2}) + \text{dist}_t(\gamma(\frac{1}{2}), x_i) \leq l - \frac{1}{4} < r.$$

So $y \in B(x_i, t_0, r)$, which confirms (2.3).

Let us now prove by induction on $k = 1, 2, \dots$ that for any $x \in \mathbf{M}$

$$(2.4) \quad |B(x, t, \frac{1}{4}k)|_t < C_*^k.$$

For $k = 1$, the inequality follows from (2.1) (assuming $c < 1$ and hence $C_* > C$). If the inequality is true for k , then we can use (2.3) to conclude

$$|B(x, t, \frac{1}{4}(k+1))|_t \leq |B(x_1, t, \frac{1}{4}k)|_t + \dots + |B(x_N, t, \frac{1}{4}k)|_t \leq N \cdot C_*^k \leq C_* \cdot C_*^k = C_*^{k+1}.$$

So (2.4) also holds for $k+1$. This finishes the proof of (2.4).

The assertion of the lemma now follows from (2.1) for $r < 1$. For $r \geq 1$ choose $k \in \mathbb{N}$ such that $\frac{1}{4}(k-1) \leq r < \frac{1}{4}k$. Then, by (2.4), we have

$$|B(x, t, r)|_t < |B(x, t, \frac{1}{4}k)|_t < C_*^k = C_* e^{(\log C_*)(k-1)} \leq C_* e^{4(\log C_*)r}.$$

This finishes the proof. \square

3. GENERALIZED MAXIMUM PRINCIPLE

Consider a Ricci flow $(g_t)_{t \in I}$ on a closed manifold \mathbf{M} . In the following we will consider the heat kernel $K(x, t; y, s)$ on a Ricci flow background. That is, for any $(y, s) \in \mathbf{M} \times I$ the kernel $K(\cdot, \cdot; y, s)$ is defined for $t > s$ and $x \in \mathbf{M}$ and satisfies

$$(\partial_t - \Delta_x)K(x, t; y, s) = 0 \quad \text{and} \quad \lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y.$$

Then, for fixed $(x, t) \in \mathbf{M} \times I$, the function $K(x, t; \cdot, \cdot)$, which is defined for $s < t$, is a kernel for the conjugate heat equation

$$(-\partial_s - \Delta_y + R(y, s))K(x, t; y, s) = 0 \quad \text{and} \quad \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x.$$

Recall that for any $s < t$ and $x \in \mathbf{M}$ we have

$$(3.1) \quad \int_{\mathbf{M}} K(x, t; y, s) dg_s(y) = 1.$$

Lemma 3.1. *Let $(\mathbf{M}, (g_t)_{t \in [0, 1]})$ be a Ricci flow on a compact manifold \mathbf{M} with $|R| \leq R_0$ on $\mathbf{M} \times [0, 1]$ for some constant $R_0 \geq 0$. Then for any $(x, t) \in \mathbf{M} \times (0, 1]$ we have*

$$\int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \leq R_0.$$

Proof. This follows from the identities

$$R(x, t) = \int_{\mathbf{M}} K(x, t; y, 0) R(y, 0) dg_0(y) + 2 \int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds$$

and (3.1) as well as $R(x, t) \leq R_0$ and $R(\cdot, 0) \geq -R_0$ on \mathbf{M} . \square

We will now use the Gaussian bounds from [BZ15] to bound the forward heat kernel in terms of the backwards conjugate heat kernel based at a certain point and time. Note that in the following Lemma we only obtain estimates on the time-interval $[0, 1]$, but we need to assume that the flow exists on $[-1, 1]$. This is due to an extra condition in [BZ15, Theorem 1.4].

Lemma 3.2. *For any $A < \infty$ there are uniform constants $C_1 = C_1(A)$, $Y = Y(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Let $0 \leq t_1 < t_2 < t_3 \leq 1$ such that

$$Y(t_2 - t_1) \leq t_3 - t_2 \leq 10Y(t_2 - t_1).$$

Then for all $x, y \in \mathbf{M}$

$$K(x, t_2; y, t_1) < C_1 K(y, t_3; x, t_2).$$

Proof. Recall that, by [BZ15, Theorem 1.4], there are constants $C_1^* = C_1^*(A)$, $C_2^* = C_2^*(A) < \infty$ such that for any $0 \leq s < t \leq 1$

$$(3.2) \quad \frac{1}{C_1^*(t-s)^{n/2}} \exp\left(-\frac{C_2^* d_s^2(x, y)}{t-s}\right) < K(x, t; y, s) < \frac{C_1^*}{(t-s)^{n/2}} \exp\left(-\frac{d_s^2(x, y)}{C_2^*(t-s)}\right).$$

Set now

$$Y := (C_2^*)^2 \quad \text{and} \quad C_1 := (C_1^*)^2 (10Y)^{n/2}.$$

Then

$$\begin{aligned}
K(x, t_2; y, t_1) &< \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^*(t_2 - t_1)}\right) \\
&\leq \frac{C_1^*}{(10Y)^{-n/2}(t_3 - t_2)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^*(t_2 - t_1)}\right) \\
&\leq C_1 \frac{1}{C_1^*(t_3 - t_2)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x, y)}{C_2^*Y^{-1}(t_3 - t_2)}\right) \\
&= C_1 \frac{1}{C_1^*(t_3 - t_2)^{n/2}} \exp\left(-\frac{C_2^*d_{t_1}^2(x, y)}{(t_3 - t_2)}\right) < C_1 K(y, t_3, x, t_2).
\end{aligned}$$

This finishes the proof. \square

Next, we combine Lemmas 3.1 and 3.2 to obtain the following bound.

Lemma 3.3. *For any $A < \infty$ there are uniform constants $C_2 = C_2(A) < \infty$, $\theta_2 = \theta_2(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Then for any $0 \leq t < 1$ and $0 < a \leq \theta_2(1 - t)$ and $x \in \mathbf{M}$ we have

$$\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds < C_2 R_0^{1/2} \sqrt{a}.$$

Proof. Choose $\theta_2 := \frac{1}{2}Y^{-1}$ and set

$$t_3 := t + 2Ya \leq 1.$$

So for any $s \in [t + a, t + 2a]$ we have

$$Y(s - t) \leq Y \cdot 2a = t_3 - t \leq 10Ya \leq 10Y(s - t).$$

So by Lemma 3.2, we have for any $(y, s) \in \mathbf{M} \times [t + a, t + 2a]$

$$K(y, s; x, t) < C_1 K(x, t_3; y, s).$$

We can then conclude, using Cauchy-Schwarz, (3.1) and Lemma 3.1, that

$$\begin{aligned}
&\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\
&\leq C_1 \int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|(y, s) dg_s(y) ds \\
&\leq C_1 \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) dg_s(y) ds \right)^{1/2} \\
&\quad \cdot \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \right)^{1/2} \\
&= C_1 \sqrt{a} \left(\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(x, t_3; y, s) |\text{Ric}|^2(y, s) dg_s(y) ds \right)^{1/2} \\
&\leq C_1 R_0^{1/2} \sqrt{a}.
\end{aligned}$$

This proves the desired result. \square

Lemma 3.4. *For any $A < \infty$ there are constants $C_3 = C_3(A) < \infty$, $\theta_3 = \theta_3(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Then for any $0 \leq s < t \leq 1$ with $t - s \leq \theta_3(1 - s)$ and any $x \in \mathbf{M}$, we have

$$\int_s^t \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds < C_3 R_0^{1/2} \sqrt{t - s}.$$

Proof. Choose $\theta_3(A) = \theta_2(A)$. Then, using Lemma 3.3,

$$\begin{aligned} \int_s^t \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) &= \sum_{k=1}^{\infty} \int_{s+(t-s)2^{-k}}^{s+2(t-s)2^{-k}} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\ &\leq \sum_{k=1}^{\infty} C_2 R_0^{1/2} \sqrt{(t-s)2^{-k}} \\ &= C_2 R_0^{1/2} \sqrt{t-s} \sum_{k=1}^{\infty} 2^{-k/2} \\ &\leq C C_2 R_0^{1/2} \sqrt{t-s}. \end{aligned}$$

This proves the desired estimate. \square

Proposition 3.5. *For every $A < \infty$ there are constants $\theta_4 = \theta_4(A) > 0$ and $C_4 = C_4(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1,1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $H > 1$ and $[t_1, t_2] \subset [0, 1)$ be a sub-interval with $t_2 - t_1 \leq \theta_4 \min\{(1 - t_1), H^{-1}\}$ and consider a non-negative function $f \in C^\infty(\mathbf{M} \times [t_1, t_2])$ that satisfies the following evolution inequality in the barrier sense:

$$-\partial_t f \leq \Delta f + H |\text{Ric}| f - R f.$$

Then

$$\max_{\mathbf{M}} f(\cdot, t_1) \leq (1 + C_4 H R_0^{1/2} \sqrt{t_2 - t_1}) \max_{\mathbf{M}} f(\cdot, t_2).$$

Note that with similar techniques, we can analyze the evolution inequality $-\partial_t f \leq \Delta f + H |\text{Ric}|^p f$ for any $p \in (0, 2)$.

Proof. We first find that that for any $(x, t) \in \mathbf{M} \times [-1, 1)$ and $t < s \leq 1$

$$\begin{aligned} \frac{d}{ds} \int_{\mathbf{M}} K(y, s; x, t) dg_s(y) &= \int_{\mathbf{M}} (\Delta_y K(y, s; x, t) - K(y, s; x, t) R(y, s)) dg_s(y) \\ &\leq R_0 \int_{\mathbf{M}} K(y, s; x, t) dg_s(y), \end{aligned}$$

which implies

$$\int_{\mathbf{M}} K(y, s; x, t) dg_s(y) \leq e^{R_0(s-t)}.$$

So for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$ we have by Lemma 3.4, assuming $\theta_4 \leq \theta_3$ and $C_3 > 1$,

$$\begin{aligned} f(x, t) &\leq \int_{\mathbf{M}} K(y, t_2; x, s) f(y, t_2) dg_{t_2}(y) \\ &\quad + \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) \cdot H |\text{Ric}|(y, s) \cdot f(y, s) dg_s(y) ds \\ &\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) |\text{Ric}|(y, s) dg_s(y) ds \\ &\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \cdot C_3 R_0^{1/2} \sqrt{t_2 - t}. \end{aligned}$$

It follows that

$$\max_{\mathbf{M} \times [t, t_2]} f \leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + \left(\max_{\mathbf{M} \times [t, t_2]} f \right) \cdot C_3 H R_0^{1/2} \sqrt{t_2 - t}.$$

So if $t_2 - t < (2C_3 H)^{-2}$, then

$$\max_{\mathbf{M} \times [t, t_2]} f \leq \frac{e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2)}{1 - C_3 H R_0^{1/2} \sqrt{t_2 - t}} \leq (1 + 10C_3 H R_0^{1/2} \sqrt{t_2 - t}) \max_{\mathbf{M}} f(\cdot, t_2).$$

This finishes the proof. \square

4. PROOF OF THEOREM 1.1

We will first establish a lower bound on the distortion of the distance:

Lemma 4.1. *For every $A < \infty$ there is a constant $C_5 = C_5(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq 1$ on $\mathbf{M} \times [-1, 1]$. Let $[t_1, t_2] \subset [0, 1]$ be a sub-interval and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \geq d_{t_1}(x_1, x_2) - C_5 \sqrt{t_2 - t_1}.$$

Proof. Set $d := d_{t_1}(x_1, x_2)$ and let $u \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times (t_1, t_2])$ be a solution to the heat equation

$$\partial_t u = \Delta u, \quad u(\cdot, t_1) = d_{t_1}(x_1, \cdot).$$

Then for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$

$$u(x, t) = \int_{\mathbf{M}} K(x, t; y, t_1) u(t_1) dg_{t_1}(y) = \int_{\mathbf{M}} K(x, t; y, t_1) d_{t_1}(x_1, y) dg_{t_1}(y).$$

Using [BZ15, Theorem 1.4] (compare also with (3.2)), we find that by Lemma 2.1

$$\begin{aligned}
u(x_1, t_2) &\leq \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1, y)}{C_2^*(t_2 - t_1)}\right) d_{t_1}(x_1, y) dg_{t_1}(y) \\
&= \sum_{k=-\infty}^{\infty} \int_{B(x_1, t_1, 2^k) \setminus B(x_1, t_1, 2^{k-1})} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1, y)}{C_2^*(t_2 - t_1)}\right) \\
&\quad \cdot d_{t_1}(x_1, y) dg_{t_1}(y) \\
&\leq \sum_{k=-\infty}^{\infty} |B(x_1, t_1, 2^k)|_{t_1} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k-2}}{C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\leq \sum_{k=-\infty}^{\infty} C_0(2^k)^n e^{C_0 2^k} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k}}{4C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\leq \int_{\mathbb{R}^n} \frac{CC_0 C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(2C_0|x| - \frac{|x|^2}{4C_2^*(t_2 - t_1)}\right) |x| dx \\
&= \sqrt{t_2 - t_1} \int_{\mathbb{R}^n} CC_0 C_1^* \exp\left(2C_0|x|\sqrt{t_2 - t_1} - \frac{|x|^2}{4C_2^*}\right) |x| dx \leq C\sqrt{t_2 - t_1}
\end{aligned}$$

On the other hand, using (3.1),

$$\begin{aligned}
|d - u(x_2, t_2)| &= \left| \int_{\mathbf{M}} K(x_2, t; y, t_1) (d - d_{t_1}(x_1, y)) dg_{t_1}(y) \right| \\
&\leq \int_{\mathbf{M}} K(x_2, t; y, t_1) |d_{t_1}(x_1, x_2) - d_{t_1}(x_1, y)| dg_{t_1}(y) \leq \int_{\mathbf{M}} K(x_2, t; y, t_1) d_{t_1}(x_2, y) dg_{t_1}(y).
\end{aligned}$$

So similarly,

$$|d - u(x_2, t_2)| \leq C\sqrt{t_2 - t_1}.$$

It follows that

$$(4.1) \quad |u(x_1, t_2) - u(x_2, t_2)| \geq d - 2C\sqrt{t_2 - t_1}.$$

Next, consider the quantity $|\nabla u|$ on $\mathbf{M} \times [t_1, t_2]$. It is not hard to check that, in the barrier sense,

$$(4.2) \quad \partial_t |\nabla u| \leq \Delta |\nabla u|.$$

Since $|\nabla u|(\cdot, t_1) \leq 1$, we have by the maximum principle that $|\nabla u| \leq 1$ on $\mathbf{M} \times [t_1, t_2]$. So

$$|u(x_1, t_2) - u(x_2, t_2)| \leq d_{t_2}(x_1, x_2).$$

Together with (4.1) this gives us

$$d_{t_2}(x_1, x_2) \geq d - 2C\sqrt{t_2 - t_1} = d_{t_1}(x_1, x_2) - 2C\sqrt{t_2 - t_1}.$$

This finishes the proof. \square

For the upper bound on the distance distortion, we will argue similarly, by reversing time. The derivation of the bound on $|\nabla u|$ will now be more complicated, since the equation (4.2) will have an extra $4|\text{Ric}||\nabla u|$ term. We will overcome this difficulty by applying the generalized maximum principle from Proposition 3.5.

Lemma 4.2. *For every $A < \infty$ there are constants $\theta_6 = \theta_6(A) > 0$ and $C_6 = C_6(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $[t_1, t_2] \subset [0, 1]$ be a sub-interval with $t_2 - t_1 \leq \theta_6(1 - t_1)$ and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \leq \exp(C_6 R_0^{1/2} \sqrt{t_2 - t_1}) d_{t_1}(x_1, x_2) + C_6 \sqrt{t_2 - t_1}.$$

Proof. Set $d := d_{t_2}(x_1, x_2)$. For $i = 1, 2$ let $u_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2])$ be a solution to the backwards (not the conjugate!) heat equation

$$(4.3) \quad -\partial_t u_i = \Delta u_i, \quad u_i(\cdot, t_2) = d_{t_2}(x_i, \cdot)$$

and let $v_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2])$ be a solution to the conjugate heat equation

$$-\partial_t v_i = \Delta v_i - R v_i, \quad v_i(\cdot, t_2) = d_{t_2}(x_i, \cdot).$$

Note that by the maximum principle, we have on $\mathbf{M} \times [t_1, t_2]$

$$(4.4) \quad u_1 + u_2 \geq \min_{\mathbf{M}} (u_1(\cdot, t_2) + u_2(\cdot, t_2)) \geq \min_{\mathbf{M}} (d_{t_2}(x_1, \cdot) + d_{t_2}(x_2, \cdot)) \geq d.$$

We also claim that we have for all $t \in [t_1, t_2]$

$$(4.5) \quad u_i(\cdot, t) \leq e^{R_0(t_2-t)} v_i(\cdot, t).$$

This inequality follows by the maximum principle and by the fact that whenever $v_i \geq 0$, we have

$$(-\partial_t - \Delta)(e^{R_0(t_2-t)} v_i(\cdot, t)) = e^{R_0(t_2-t)} R_0 v_i(\cdot, t) - e^{R_0(t_2-t)} R(\cdot, t) v_i(\cdot, t) \geq 0.$$

We now make use of the fact that for any $x \in \mathbf{M}$,

$$v_i(x, t_1) = \int_{\mathbf{M}} K(y, t_2; x, t_1) v_i(y, t_2) dg_{t_2}(y) = \int_{\mathbf{M}} K(y, t_2; x, t_1) d_{t_2}(x_i, y) dg_{t_2}(y)$$

and

$$K(y, t_2; x, t_1) < \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x, y)}{C_2^*(t_2 - t_1)}\right),$$

for some constants C_1^*, C_2^* , which depend only on A . Note that the latter inequality is similar to (3.2) except that the distance between x, y is taken at time t_2 . This inequality follows from [BZ15, Theorem 1.4] and the subsequent comment in that paper. We can hence estimate, similarly as in the proof of Lemma 4.1,

$$v_i(x_i, t_1) \leq \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x_i, y)}{C_2^*(t_2 - t_1)}\right) d_{t_2}(x_i, y) dg_{t_2}(y) \leq C \sqrt{t_2 - t_1}.$$

So, using (4.5), we have

$$u_i(x_i, t_1) \leq C e^{R_0(t_2-t_1)} \sqrt{t_2 - t_1} \leq 10C \sqrt{t_2 - t_1}.$$

So by (4.4) we have

$$u_1(x_2, t_1) \geq d - u_2(x_2, t_1) \geq d - 10C \sqrt{t_2 - t_1}.$$

This implies

$$(4.6) \quad |u_1(x_1, t_1) - u_1(x_2, t_2)| \geq d - 20C \sqrt{t_2 - t_1}.$$

Taking derivatives of (4.3), we obtain the evolution inequality

$$-\partial_t |\nabla u_1| \leq \Delta |\nabla u_1| + 4|\text{Ric}| \cdot |\nabla u_1| \leq \Delta |\nabla u_1| + (4 + \sqrt{n})|\text{Ric}| \cdot |\nabla u_1| - R|\nabla u_1|,$$

which holds in the barrier sense. Note that by definition $|\nabla u_1(\cdot, t_2)| \leq 1$. So, by Proposition 3.5, we have for sufficiently small θ_6

$$|\nabla u_1(\cdot, t_1)| \leq 1 + CR_0^{1/2} \sqrt{t_2 - t_1}.$$

So, using (4.6), we obtain

$$\begin{aligned} d_{t_2}(x_1, x_2) - 10C\sqrt{t_2 - t_1} &\leq |u(x_1, t_1) - u(x_2, t_2)| \\ &\leq (1 + CR_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x_1, x_2) \leq \exp(CR_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x_1, x_2). \end{aligned}$$

This finishes the proof. \square

Next, we remove the assumption $t_2 - t_1 \leq \theta_6(1 - t_1)$ from Lemma 4.2.

Lemma 4.3. *For every $A < \infty$ there is a constant $C_7 = C_7(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t \in [-1, 1]})$ be a Ricci flow on a compact, n -dimensional manifold \mathbf{M} with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $0 \leq t_1 \leq t_2 \leq 1$ and consider two points $x, y \in \mathbf{M}$. Then

$$d_{t_2}(x, y) \leq \exp(C_7 R_0^{1/2} \sqrt{t_2 - t_1})d_{t_1}(x, y) + C_7 \sqrt{t_2 - t_1}.$$

Proof. In the case in which $t_2 - t_1 \leq \theta_6(1 - t_1)$, the bound follows immediately from Lemma 4.2. Let us now assume that $t_2 - t_1 > \theta_6(1 - t_1)$. By continuity we may also assume without loss of generality that $t_2 < 1$.

Choose times

$$t'_k := 1 - (1 - \theta_6)^k(1 - t_1)$$

and observe that $t'_0 = t_1$ and

$$t'_{k+1} - t'_k = \theta_6(1 - \theta_6)^k(1 - t_1) = \theta_6(1 - t'_k).$$

So by Lemma 4.2

$$\begin{aligned} d_{t'_k}(x, y) &\leq \exp\left(C_6 R_0^{1/2} \sum_{l=1}^k \sqrt{t'_l - t'_{l-1}}\right) d_{t_1}(x, y) \\ &\quad + C_6 \sum_{l=1}^k \exp\left(C_6 R_0^{1/2} \sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}}\right) \sqrt{t'_l - t'_{l-1}}. \end{aligned}$$

Since

$$\sum_{l=1}^k \sqrt{t'_l - t'_{l-1}} = \sum_{l=1}^k \sqrt{\theta_6(1 - \theta_6)^{l/2} \sqrt{1 - t_1}} \leq C' \sqrt{1 - t_1}$$

and

$$\begin{aligned} \sum_{l=1}^k \exp \left(C_6 R_0^{1/2} \sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}} \right) \sqrt{t'_l - t'_{l-1}} \\ \leq \sum_{l=1}^k \exp \left(C_6 C' R_0^{1/2} \sqrt{1 - t_1} \right) \sqrt{t'_l - t'_{l-1}} \leq C'' \sqrt{1 - t_1}, \end{aligned}$$

we find that for a generic constant $C < \infty$

$$d_{t'_k}(x, y) \leq \exp \left(C R_0^{1/2} \sqrt{1 - t_1} \right) d_{t_1}(x, y) + C \sqrt{1 - t_1}.$$

Choose now k such that $t'_k \leq t_2 < t'_{k+1}$. Then $t_2 - t'_k \leq t'_{k+1} - t'_k \leq \theta_6(1 - t'_1)$, so again by Lemma 4.2, we have

$$\begin{aligned} d_{t_2}(x, y) &\leq \exp \left(C_6 R_0^{1/2} \sqrt{t_2 - t'_k} \right) d_{t'_k}(x, y) + C_6 \sqrt{t_2 - t'_k} \\ &\leq \exp \left((C + C_6) R_0^{1/2} \sqrt{1 - t_1} \right) d_{t_1}(x, y) + C \exp(1 + C_6) \sqrt{1 - t_1} + C_6 \sqrt{1 - t_1}. \end{aligned}$$

The claim now follows using $\sqrt{1 - t_1} < \theta_6^{-1/2} \sqrt{t_2 - t_1}$. \square

We can finally prove Theorem 1.1.

Proof of Theorem 1.1. Consider the Ricci flow $(\mathbf{M}^n, (g_t)_{t \in [0,1]})$ with $\nu[g_0, 1 + A^{-1}] \geq -A$ and $|R| \leq R_0$ for $0 \leq R_0 \leq A$. After replacing A by $4A + 2$, we may assume without loss of generality that $A > 2$ and that we even have $\nu[g_0, 1 + 4A^{-1}] \geq -A$.

We will first prove the distance bounds for the case in which $t_1 > 0$ and $t_2 \leq (1 + A^{-1})t_1$. By monotonicity of ν (compare with [BZ15, sec 2]), we find that for any $t \in [0, 1]$ we have

$$\nu[g_t, 4A^{-1}] \geq \nu[g_0, 1 + 4A^{-1}] \geq -A.$$

Restrict the flow to the time-interval $[(1 - A^{-1})t_1, (1 + A^{-1})t_1]$ and parabolically rescale by $A^{1/2}t_1^{-1/2}$ to obtain a flow $(\tilde{g}_t)_{t \in [A^{-1}, A+1]}$. Then $\nu[\tilde{g}_{A^{-1}}, 4] \geq -A$ and $|\tilde{R}| \leq \tilde{R}_0 := A^{-1}t_1 R_0 \leq 1$. Then t_1, t_2 correspond to times $\tilde{t}_1 := A, \tilde{t}_2 := At_1^{-1}t_2$ and we have

$$\tilde{R}_0^{1/2} \sqrt{\tilde{t}_2 - \tilde{t}_1} = R_0^{1/2} \sqrt{t_2 - t_1}.$$

So the distance bounds follow from Lemmas 4.1 and 4.3.

Consider now the case in which $t_2 > (1 + A^{-1})t_1$. So $t_1 < \lambda t_2$, where $\lambda := (1 + A^{-1})^{-1} < 1$. By continuity we may assume without loss of generality that $t_1 > 0$. Then we can find $1 \leq k_2 < k_1$ such that $t_1 \in [\lambda^{k_1}, \lambda^{k_1-1}]$ and $t_2 \in [\lambda^{k_2}, \lambda^{k_2-1}]$. Using our previous conclusions, we find

$$d_{t_2}(x, y) \geq d_{\lambda^{k_2}}(x, y) - C \sqrt{\lambda^{k_2}} \geq \dots \geq d_{t_1}(x, y) - C \sum_{l=k_1}^{k_2} \sqrt{\lambda^l} \geq d_{t_1}(x, y) - C' C \lambda^{k_2/2}.$$

Since $t_1 < \lambda t_2$, we have $\sqrt{t_2 - t_1} > \sqrt{(1 - \lambda)t_2} > \sqrt{1 - \lambda} \sqrt{\lambda^{k_2}}$. So

$$d_{t_2}(x, y) \geq d_{t_1}(x, y) - C' C (1 - \lambda)^{-1/2} \sqrt{t_2 - t_1}.$$

This establishes the lower bound.

For the upper bound, set $t'_0 := t_1$, $t'_1 := \lambda^{k_1-1}$, \dots , $t'_{k_1-k_2} := \lambda^{k_2}$, $t'_{k_1-k_2+1} := t_2$. Then we have by our previous conclusions

$$\begin{aligned} d_{t_2}(x, y) &\leq \exp\left(CR_0^{1/2} \sum_{l=1}^{k_1-k_2+1} \sqrt{t'_l - t'_{l-1}}\right) d_{t_1}(x, y) \\ &\quad + C \sum_{l=1}^{k_2-k_1+1} \exp\left(CR_0^{1/2} \sum_{j=l+1}^{k_1-k_2+1} \sqrt{t'_j - t'_{j-1}}\right) \sqrt{t'_l - t'_{l-1}} \end{aligned}$$

Similarly as in the proof of Lemma 4.3, we conclude

$$d_{t_2}(x, y) \leq \exp\left(CR_0^{1/2} \sqrt{\lambda^{k_2}}\right) d_{t_1}(x, y) + C\sqrt{\lambda^{k_2}}.$$

Again, using $\sqrt{t_2 - t_1} > \sqrt{1 - \lambda} \sqrt{\lambda^{k_2}}$, we get the desired bound. \square

5. PROOF OF COROLLARY 1.3

Proof of Corollary 1.3. For each i consider the metric \bar{d}^i on \mathbf{M}^i with

$$\bar{d}^i(x, y) := \int_0^1 d_t^i(x, y) dt.$$

Note that by the Hölder bound in Theorem 1.1 there is a uniform constant $c' > 0$ such that for all $t, t' \in [0, 1]$ we have $d_{t'}^i(x, y) > \frac{1}{2} d_t^i(x, y)$ whenever $|t - t'| \leq c'(d_t^i(x, y))^2$. So there is a uniform constant $c > 0$ such that for all $t \in [0, 1]$

$$(5.1) \quad \bar{d}^i(x, y) \geq c(\min\{d_t^i(x, y), 1\})^3.$$

So by the triangle inequality and Theorem 1.1, for any $A < \infty$ there is a constant $C < \infty$ such that for any $x, y, x', y' \in \mathbf{M}$ and $t, t' \in [0, 1]$ with $\bar{d}^i(x, y) + \bar{d}^i(x, x') + \bar{d}^i(y, y') < A$ we have

$$(5.2) \quad |d_t^i(x, y) - d_{t'}^i(x', y')| \leq C(\bar{d}^i(x, x'))^{1/3} + C(\bar{d}^i(y, y'))^{1/3} + C|t - t'|^{1/2}.$$

We first argue that the sequence $(\mathbf{M}^i, \bar{d}^i)$ is uniformly totally bounded in the following sense: For any $0 < a < b$ there is a number $N = N(a, b) < \infty$ such that for any i and any $x \in \mathbf{M}^i$, the ball $\bar{B}^i(x, b) := \{x \in \mathbf{M}^i : \bar{d}^i(x, z) < b\}$ contains at most N pairwise disjoint balls $\bar{B}^i(y_j, a)$, $j = 1, \dots, m$. Fix $0 < a < b$ and assume without loss of generality that $a < 1$. By (5.1) there is a constant $b' = b'(b) < \infty$ such that $\bar{B}^i(x, b) \subset B^i(x, t, b')$ for all $t \in [0, 1]$.

Assume that $y_1, \dots, y_m \in \bar{B}^i(x, b)$ such that the balls $\bar{B}^i(y_j, a)$ are pairwise disjoint. This implies $\bar{d}^i(y_{j_1}, y_{j_2}) \geq 2a$ for all $j_1 \neq j_2$. By the Hölder bound in Theorem 1.1, we may find a large integer $L = L(a) < \infty$ such that whenever $\bar{d}^i(y, y') \geq 2a$ for some points $y, y' \in \mathbf{M}^i$, then $d_{\frac{t}{L}}^i(y, y') > a$ for some $l \in \{1, \dots, L\}$. So for any $j_1 \neq j_2$, there is an $l_{j_1, j_2} \in \{1, \dots, L\}$ such that

$$d_{\frac{l_{j_1, j_2}}{L}}^i(y_{j_1}, y_{j_2}) > a.$$

This implies the following statement: If we form the L -fold Cartesian product $\mathbf{M}^{i, L} := (\mathbf{M}^i)^L = \mathbf{M} \times \dots \times \mathbf{M}$ equipped with the metric $g_{\frac{1}{L}}^i \oplus \dots \oplus g_{\frac{L-1}{L}}^i$ and if we define $y_j^L :=$

$(y_j, \dots, y_j) \in \mathbf{M}^{i,L}$, then $d^{\mathbf{M}^{i,L}}(y_{j_1}^L, y_{j_2}^L) > a$ for any $j_1 \neq j_2$. So the $\frac{1}{2}a$ -balls around $y_{j_1}^L$ are pairwise disjoint and contained in $B^i(x, \frac{1}{L}, b' + a) \times \dots \times B^i(x, \frac{L-1}{L}, b' + a)$. Using (2.1) and Lemma 2.1, we conclude that

$$\left(c \left(\frac{a}{\sqrt{L}}\right)^n\right)^L \cdot m \leq \left(C_0(b')^n e^{C_0 b'}\right)^L,$$

which yields an upper bound on m . So the sequence $(\mathbf{M}^i, \bar{d}^i)$ is in fact uniformly totally bounded.

We may now pass to a subsequence and assume that $(\mathbf{M}^i, \bar{d}^i, x_i)$ converges to some metric space $(\bar{\mathbf{M}}, \bar{d}, \bar{x})$ in the pointed Gromov-Hausdorff sense. By (5.2) and Arzelà-Ascoli and after passing to another subsequence, the sequence of time-dependent metrics $(d^i)_{t \in [0,1]}$ converges locally uniformly to a time-dependent, continuous family of pseudo-metrics $(d_t^\infty)_{t \in [0,1]}$ on $\bar{\mathbf{M}}$. So for any $t \in [0,1]$, the pointed metric spaces $(\mathbf{M}^i, d_t^i, x_i)$ converge to $(\bar{\mathbf{M}}/\sim_t, d_t^\infty, \bar{x})$ in the pointed Gromov-Hausdorff sense. Passing to another subsequence once again, and using (2.1), we can ensure that also the volume forms dg_t^i converge uniformly for every rational $t \in [0,1]$. Since $e^{-A|t_2-t_1|} dg_{t_1}^i \leq dg_{t_2}^i \leq e^{A|t_2-t_1|} dg_{t_1}^i$, the convergence holds for any $t \in [0,1]$. The doubling property for balls of radius less than \sqrt{t} follows from (2.1) after parabolic rescaling by $(\frac{1}{2}t)^{-1/2}$. \square

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DEPARTMENT OF MATHEMATICS, UC BERKELEY, BERKELEY, CA 94720, USA
E-mail address: rbamler@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521, USA
E-mail address: qizhang@math.ucr.edu